

# On the generators of the canonical module of a Hibi ring: a criterion of level property and the degrees of generators

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## Abstract

In this paper, we study the minimal generating system of the canonical module of a Hibi ring. Using the results, we state a characterization of a Hibi ring to be level. We also give a characterization of a Hibi ring to be of type 2. Further, we show that the degrees of the elements of the minimal generating system of the canonical module of a Hibi ring form a set of consecutive integers.

Hibi ring, canonical module, level ring, Cohen-Macaulay type

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## 1 Introduction

A Hibi ring is an algebra with straightening law on a distributive lattice with many good properties. It appears in many scenes of the study of rings with combinatorial structure. In many important cases, the initial subalgebra of a subalgebra of a polynomial ring has a structure of a Hibi ring. A Hibi ring is a normal affine semigroup ring and is therefore Cohen-Macaulay by the result of Hochster [2]. Moreover there is a good description of the canonical module of it [1]. However, as the Example of [1, §1 e)] shows, a Hibi ring is not necessarily level.

The present author gave a sufficient condition for a Hibi ring to be level [3] and showed that the homogeneous coordinate ring of a Schubert subvariety

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of a Grassmannian is level. However, the sufficient condition given in [3] is far from necessary.

In this paper we analyze the generating system of the canonical module of a Hibi ring by using the description of the canonical module of a normal affine semigroup ring by Stanley [5]. Using this, we give a combinatorial necessary and sufficient condition for a Hibi ring to be level. We also show that the set of degrees of the generators of the canonical module of a Hibi ring form a set of consecutive integers. Here we call a member of the minimal generating system a generator for simplicity.

Further, we give a combinatorial criterion that the Hibi ring under consideration has Cohen-Macaulay type 2. Recall that a Hibi ring is Gorenstein i.e., type of it is 1 if and only if the set of join-irreducible elements of the distributive lattice defining that Hibi ring is pure [1, §3 d)]. Since type 2 is next to type 1, we think it worth to give a criterion of a Hibi ring has type 2.

This paper is organized as follows. Section 2 is a preparation for the main argument. In Section 2, we recall some basic facts on Hibi rings, normal affine semigroup rings and levelness of standard graded algebras. Also, we state some basic facts on posets.

In Section 3, we state a necessary and sufficient condition that a Hibi ring is level by considering the degrees of the generators of the canonical module of the Hibi ring. The key notion is a sequence with “condition N” (see Definition 3.1). We named the condition as condition N from the shape of the Hasse diagram related to the elements in the sequence. We show that from a generator of the canonical module of a Hibi ring, one can construct a sequence with condition N with additional condition in the poset of join-irreducible elements corresponding to the Hibi ring. We also show a kind of converse of this fact. Thus, the range of the degrees of the generators of the canonical module of a Hibi ring can be described by sequences with condition N. Using this, we obtain a combinatorial criterion for a Hibi ring to be level.

Further, we show that the degrees of the elements of the minimal generating system of the canonical module of a Hibi ring form a consecutive integers, i.e., if there are elements of degrees  $d_1$  and  $d_2$  in the minimal system of generators of the canonical module of a Hibi ring with  $d_1 < d_2$ , then there exists an element with degree  $d$  in the minimal system of the generators of the canonical module of the Hibi ring for any integer  $d$  with  $d_1 \leq d \leq d_2$ .

In Sections 4 and 5, we state a combinatorial criterion that a Hibi ring is of type 2. In Section 4, we state a necessary and sufficient condition that a Hibi ring is level and has type 2 and in Section 5, we state a necessary and sufficient condition that a Hibi ring is not level and has type 2.

## 2 Preliminaries

In this paper, all rings and algebras are assumed to be commutative with identity element. We denote by  $\mathbf{N}$  the set of non-negative integers, by  $\mathbf{Z}$  the set of integers, by  $\mathbf{R}$  the set of real numbers and by  $\mathbf{R}_{\geq 0}$  the set of non-negative real numbers.

First we recall some definitions concerning finite partially ordered sets (poset for short).

**Definition** Let  $Q$  be a finite poset.

- A chain in  $Q$  is a totally ordered subset of  $Q$ .
- For a chain  $X$  in  $Q$ , we define the length of  $X$  as  $\#X - 1$ , where  $\#X$  denotes the cardinality of  $X$ .
- The maximum length of chains in  $Q$  is called the rank of  $Q$  and denoted as  $\text{rank}Q$ .
- If every maximal chain of  $Q$  has the same length, we say that  $Q$  is pure.
- If  $x, y \in Q$ ,  $x < y$  and there is no  $z \in Q$  with  $x < z < y$ , we say that  $y$  covers  $x$  and denote  $x < y$  or  $y > x$ .
- For  $x, y \in Q$  with  $x \leq y$ , we set  $[x, y]_Q := \{z \in Q \mid x \leq z \leq y\}$ , and for  $x, y \in Q$  with  $x < y$ , we set  $[x, y)_Q := \{z \in Q \mid x \leq z < y\}$ ,  $(x, y]_Q := \{z \in Q \mid x < z \leq y\}$  and  $(x, y)_Q := \{z \in Q \mid x < z < y\}$ . We denote  $[x, y]_Q$  ( $[x, y)_Q$ ,  $(x, y]_Q$  ( $(x, y)_Q$  resp.) as  $[x, y]$  ( $[x, y)$ ,  $(x, y]$ ,  $(x, y)$  resp.) if there is no fear of confusion.
- Let  $\infty$  be a new element which is not contained in  $Q$ . We set  $Q^+ := Q \cup \{\infty\}$  with the order  $x < \infty$  for any  $x \in Q$ .
- If  $I \subset Q$  and  $x \in I$ ,  $y \in Q$ ,  $y \leq x \Rightarrow y \in I$ , then we say that  $I$  is a poset ideal of  $Q$ .

Next we recall the definition and basic facts of Hibi rings. Let  $H$  be a finite distributive lattice and  $x_0$  the unique minimal element of  $H$ . Let  $P$  be the set of join-irreducible elements of  $H$ , i.e.,  $P = \{\alpha \in H \mid \alpha = \beta \vee \gamma \Rightarrow \alpha = \beta \text{ or } \alpha = \gamma\}$ . Note that we treat  $x_0$  as a join-irreducible element of  $H$ .

Then it is known that  $H$  is isomorphic to  $J(P) \setminus \{\emptyset\}$ , ordered by inclusion, by the correspondence

$$\begin{aligned} \alpha &\mapsto \{x \in P \mid x \leq \alpha \text{ in } H\} && \text{for } \alpha \in H \text{ and} \\ I &\mapsto \bigvee_{x \in I} x && \text{for } I \in J(P) \setminus \{\emptyset\}, \end{aligned}$$

where  $J(P)$  is the set of poset ideals of  $P$ . In particular, if  $P$  is a poset with unique minimal element, then there is a finite distributive lattice whose set of join-irreducible elements is  $P$ .

Let  $\{T_x\}_{x \in P}$  be a family of indeterminates indexed by  $P$  and  $K$  a field.

**Definition ([1])**  $\mathcal{R}_K(H) := K[\prod_{x \leq \alpha} T_x \mid \alpha \in H]$ .

$\mathcal{R}_K(H)$  is called the Hibi ring over  $K$  on  $H$ . We set  $\deg T_{x_0} = 1$  and  $\deg T_x = 0$  for any  $x \in P \setminus \{x_0\}$ . Then,  $\mathcal{R}_K(H)$  is a standard graded  $K$ -algebra, i.e., a non-negatively graded  $K$ -algebra whose degree 0 part is  $K$  and is generated over  $K$  by the degree 1 elements.

**Definition ([1])** For a map  $\nu: P \rightarrow \mathbf{N}$ , we set  $T^\nu := \prod_{x \in P} T_x^{\nu(x)}$ . We set  $\overline{\mathcal{T}}(P) := \{\nu: P \rightarrow \mathbf{N} \mid x \leq y \Rightarrow \nu(x) \geq \nu(y)\}$  and  $\mathcal{T}(P) := \{\nu: P \rightarrow \mathbf{N} \mid \nu(z) > 0 \text{ for any } z \in P \text{ and } x < y \Rightarrow \nu(x) > \nu(y)\}$ . For  $\nu \in \overline{\mathcal{T}}(P)$ , we extend  $\nu$  as a map from  $P^+$  to  $\mathbf{N}$  by setting  $\nu(\infty) = 0$ .

With this notation,

**Theorem 2.1 ([1])**  $\mathcal{R}_K(H) = \bigoplus_{\nu \in \overline{\mathcal{T}}(P)} KT^\nu$ . In particular, by the result of Hochster [2],  $\mathcal{R}_K(H)$  is a normal affine semigroup ring and is Cohen-Macaulay.

Note that  $\deg T^\nu = \nu(x_0)$ .

Here we recall the description of the canonical module of a normal affine semigroup ring by Stanley.

**Theorem 2.2 ([5, p. 81])** Let  $S$  be a finitely generated additive submonoid of  $\mathbf{N}^n$  and  $X_1, \dots, X_n$  indeterminates. If the affine semigroup ring  $\bigoplus_{s \in S} KX^s$  in  $K[X_1, \dots, X_n]$  is normal, then the canonical module of  $\bigoplus_{s \in S} KX^s$  is  $\bigoplus_{s \in S \cap \text{relint} \mathbf{R}_{\geq 0} S} KX^s$ , where  $\text{relint} Q$  denotes the interior of  $Q$  in the affine space spanned by  $Q$ .

By this result, the canonical module of a normal affine semigroup ring has the unique minimal fine graded generating system up to non-zero scalar multiplication. Therefore, we call a member of the minimal fine graded generating system a generator for simplicity.

By applying this theorem to  $\mathcal{R}_K(H)$ , we see the following

**Corollary 2.3** The canonical module of  $\mathcal{R}_K(H)$  is  $\bigoplus_{\nu \in \mathcal{T}(P)} KT^\nu$ .

In order to describe the generators of the canonical module of  $\mathcal{R}_K(H)$ , we state the following

**Definition** We define the order on  $\mathcal{T}(P)$  by setting  $\nu \leq \nu' \stackrel{\text{def}}{\iff} \nu' - \nu \in \overline{\mathcal{T}}(P)$  for  $\nu, \nu' \in \mathcal{T}(P)$ , where  $(\nu' - \nu)(x) := \nu'(x) - \nu(x)$ .

Then the following fact is easily verified.

**Corollary 2.4** *Let  $\nu$  be an element of  $\mathcal{T}(P)$ . Then  $T^\nu$  is a generator of the canonical module of  $\mathcal{R}_K(H)$  if and only if  $\nu$  is a minimal element of  $\mathcal{T}(P)$ .*

Finally, we recall the following characterization of Gorenstein property of  $\mathcal{R}_K(H)$  by Hibi.

**Theorem 2.5** ([1, §3 d])  *$\mathcal{R}_K(H)$  is Gorenstein if and only if  $P$  is pure.*

Here we recall that Stanley [4] defined the level property for a standard graded Cohen-Macaulay algebra over a field and showed that a standard graded Cohen-Macaulay algebra  $A$  over a field is level if and only if the degree of the generators of the canonical module of  $A$  is constant.

In the following, let  $H$  be a finite distributive lattice with unique minimal element  $x_0$  and let  $P$  be the set of join-irreducible elements of  $H$ . We set  $r := \text{rank} P^+$ . Since  $\deg T^\nu = \nu(x_0)$  for  $\nu \in \overline{\mathcal{T}}(P)$ , and  $\nu_0: P \rightarrow \mathbf{N}$ ,  $x \mapsto \text{rank}[x, \infty]$  is a minimal element of  $\mathcal{T}(P)$ , we see by Corollary 2.4 the following

**Corollary 2.6**  *$\mathcal{R}_K(H)$  is level if and only if  $\nu(x_0) = r$  for any minimal element  $\nu$  of  $\mathcal{T}(P)$ .*

Next we state some basic facts on  $\mathcal{T}(P)$ .

**Lemma 2.7** *Let  $I$  be a non-empty poset ideal of  $P$  and  $\nu \in \mathcal{T}(P)$ . Suppose that  $\nu(x) - \nu(y) \geq 2$  for any  $x \in I$  and  $y \in P^+ \setminus I$  with  $x < y$ . If we set*

$$\nu'(x) = \begin{cases} \nu(x) & \text{if } x \notin I \text{ and} \\ \nu(x) - 1 & \text{if } x \in I, \end{cases}$$

*then  $\nu' \in \mathcal{T}(P)$ . In particular,  $\nu$  is not a minimal element of  $\mathcal{T}(P)$ .*

As a corollary, we have the following

**Lemma 2.8** *Let  $\nu \in \mathcal{T}(P)$ . If  $\{z \in P^+ \mid \nu(z) > \text{rank}[z, \infty]\}$  is a non-empty poset ideal of  $P^+$ , then  $\nu$  is not a minimal element of  $\mathcal{T}(P)$ .*

We also state the following fact.

**Lemma 2.9** *Let  $Q$  be a poset and  $s$  a positive integer. If for any  $z_0, z_1, z_2$  and  $z_3 \in Q$  such that  $z_0 \leq z_1 \leq z_2 \leq z_3$ ,  $z_0$  is a minimal element of  $Q$  and  $z_3$  is a maximal element of  $Q$ , it holds*

$$\text{rank}[z_0, z_1] + \text{rank}[z_1, z_2] + \text{rank}[z_2, z_3] = s,$$

*then  $Q$  is pure of rank  $s$ .*

**Proof** Let  $x_0 < x_1 < \cdots < x_t$  be an arbitrary maximal chain of  $Q$ . We shall show that  $t = s$ .

Since

$$\text{rank}[x_0, x_t] = \text{rank}[x_0, x_0] + \text{rank}[x_0, x_0] + \text{rank}[x_0, x_t] = s$$

by assumption, we see that

$$t \leq \text{rank}[x_0, x_t] = s.$$

Assume  $t < s$ . Then  $\{i \mid i < \text{rank}[x_0, x_i]\} \neq \emptyset$ . Set  $j = \min\{i \mid i < \text{rank}[x_0, x_i]\}$ . Then  $j \geq 1$ . Further, we see by assumption, that

$$\begin{aligned} s &= \text{rank}[x_0, x_{j-1}] + \text{rank}[x_{j-1}, x_j] + \text{rank}[x_j, x_t] \\ &= (j-1) + 1 + \text{rank}[x_j, x_t]. \end{aligned}$$

Since  $\text{rank}[x_0, x_j] + \text{rank}[x_j, x_t] = \text{rank}[x_0, x_j] + \text{rank}[x_j, x_j] + \text{rank}[x_j, x_t] = s$ , we see that  $\text{rank}[x_j, x_t] = s - \text{rank}[x_0, x_j]$ . Thus,

$$s = j + (s - \text{rank}[x_0, x_j]) < j + s - j = s,$$

by the definition of  $j$ . This is a contradiction. ■

### 3 A criterion of levelness of a Hibi ring

In this section, we give a necessary and sufficient condition for a Hibi ring to be level. Recall that  $H$  is a finite distributive lattice with unique minimal element  $x_0$ ,  $P$  is the set of join-irreducible elements of  $H$  and  $r = \text{rank} P^+$ . First we make the following

**Definition 3.1** Let  $y_1, x_1, y_2, x_2, \dots, y_t, x_t$  be a (possibly empty) sequence of elements in  $P$ . We say the sequence  $y_1, x_1, y_2, x_2, \dots, y_t, x_t$  satisfies the condition N if

- (1)  $x_1 \neq x_0$ ,

- (2)  $y_1 > x_1 < y_2 > x_2 < \cdots < y_t > x_t$  and
- (3) for any  $i, j$  with  $1 \leq i < j \leq t$ ,  $y_i \not\geq x_j$ .

**Remark** A sequence with condition N may be an empty sequence, i.e.,  $t$  may be 0.

For a sequence with condition N, we make the following

**Definition** Let  $y_1, x_1, y_2, x_2, \dots, y_t, x_t$  be a sequence with condition N. We set

$$r(y_1, x_1, \dots, y_t, x_t) := \sum_{i=1}^t (\text{rank}[x_{i-1}, y_i] - \text{rank}[x_i, y_i]) + \text{rank}[x_t, \infty],$$

where we set an empty sum to be 0.

**Remark** For an empty sequence, we set  $r() = \text{rank}[x_0, \infty] = r$ .

Next we state a useful criterion of minimal property of an element of  $\mathcal{T}(P)$ . First we state the following

**Lemma 3.2** *Suppose that  $\nu \in \mathcal{T}(P)$ . If there are elements  $z_1, w_1, z_2, w_2, \dots, z_s, w_s \in P$  such that*

- (1)  $z_1 > w_1 < z_2 > w_2 < \cdots < z_s > w_s$  and
- (2)  $\nu(w_i) - \nu(z_{i+1}) = \text{rank}[w_i, z_{i+1}]$  for  $0 \leq i \leq s$ , where we set  $w_0 = x_0$  and  $z_{s+1} = \infty$ ,

*then  $\nu$  is a minimal element of  $\mathcal{T}(P)$ .*

**Remark**  $s$  may be 0 in Lemma 3.2, i.e., if  $\nu(x_0) = r$ , then  $\nu$  is a minimal element of  $\mathcal{T}(P)$ .

**Proof of Lemma 3.2** The case where  $s = 0$  is obvious. Thus, we consider the case where  $s > 0$ . Assume the contrary and suppose that there is  $\nu' \in \mathcal{T}(P)$  with  $\nu' < \nu$ . Then  $\nu(w_0) - \nu'(w_0) = (\nu - \nu')(w_0) > 0$  since  $\nu - \nu' \in \overline{\mathcal{T}}(P)$  and  $\nu \neq \nu'$ . Set  $j = \max\{i \mid \nu'(w_i) < \nu(w_i)\}$ . Since  $\nu(w_j) - \nu(z_{j+1}) = \text{rank}[w_j, z_{j+1}] \leq \nu'(w_j) - \nu'(z_{j+1})$  by assumption, we see that

$$\nu'(z_{j+1}) < \nu(z_{j+1}).$$

In particular,  $j < s$ . Since  $\nu - \nu' \in \overline{\mathcal{T}}(P)$ ,  $(\nu - \nu')(w_{j+1}) \geq (\nu - \nu')(z_{j+1})$ , i.e.,  $\nu(w_{j+1}) - \nu'(w_{j+1}) \geq \nu(z_{j+1}) - \nu'(z_{j+1}) > 0$ . This contradicts to the definition of  $j$ . ■

Next we show a kind of converse of this lemma.

**Lemma 3.3** *Let  $\nu$  be a minimal element of  $\mathcal{T}(P)$ . Then there exists a sequence of elements  $y_1, x_1, \dots, y_t, x_t$  with condition N such that*

$$\nu(x_i) - \nu(y_{i+1}) = \text{rank}[x_i, y_{i+1}]$$

for  $0 \leq i \leq t$ , where we set  $y_{t+1} = \infty$ .

**Remark**  $y_1, x_1, \dots, y_t, x_t$  may be an empty sequence, i.e.,  $t$  may be 0.

**Proof of Lemma 3.3** Set  $U_1 := \{y \in P^+ \mid \text{rank}[x_0, y]_{P^+} = \nu(x_0) - \nu(y)\}$ ,  $D_1 := \{x \in P^+ \setminus U_1 \mid \exists y \in U_1 \text{ such that } y > x\}$ ,  $U_2 := \{y \in P^+ \setminus (U_1 \cup D_1) \mid \exists x \in D_1 \text{ such that } x < y \text{ and } \text{rank}[x, y]_{P^+} = \nu(x) - \nu(y)\}$ ,  $D_2 := \{x \in P^+ \setminus (U_1 \cup D_1 \cup U_2) \mid \exists y \in U_2 \text{ such that } y > x\}$  and so on. Since  $P$  is a finite set, the procedure stops after finite steps, i.e.,  $U_t = \emptyset$  or  $D_t = \emptyset$  for some  $t$ .

It is proved by the induction on  $s$  that

$$U_1 \cup D_1 \cup U_2 \cup D_2 \cup \dots \cup U_s \cup D_s$$

is a poset ideal of  $P^+$  for any  $s$ . Therefore,

$$I = U_1 \cup D_1 \cup U_2 \cup D_2 \cup \dots$$

is a poset ideal of  $P^+$ .

Suppose that  $\infty \notin I$ . Then  $P^+ \setminus I \neq \emptyset$ . Moreover, if  $z \in I$ ,  $z' \in P^+ \setminus I$  and  $z < z'$ , then  $\nu(z) \geq \nu(z') + 2$ . In fact, if  $z \in D_s$  for some  $s$ , then  $\nu(z) - \nu(z') > \text{rank}[z, z'] \geq 1$ , since  $z' \notin U_{s+1} \cup D_s \cup U_s \cup \dots \cup U_1$ . If  $z \in U_s$  for some  $s$ , then there is  $x \in D_{s-1}$  such that  $\nu(x) - \nu(z) = \text{rank}[x, z]$ , where we set  $D_0 = \{x_0\}$ . Since  $z' \notin U_s \cup D_{s-1} \cup U_{s-1} \cup \dots \cup U_1$ , we see that  $\nu(x) - \nu(z') > \text{rank}[x, z']$ . Therefore,

$$\begin{aligned} \nu(z) - \nu(z') &= (\nu(x) - \nu(z')) - (\nu(x) - \nu(z)) \\ &> \text{rank}[x, z'] - \text{rank}[x, z] \geq \text{rank}[z, z'] \geq 1. \end{aligned}$$

Thus, by Lemma 2.7, we see that  $\nu$  is not a minimal element of  $\mathcal{T}(P)$ , contradicts the assumption.

Therefore,  $\infty \in I$ . Take  $t$  with  $\infty \in U_{t+1}$ . Set  $y_{t+1} = \infty$ , take  $x_t \in D_t$  such that  $x_t < y_{t+1}$  and  $\text{rank}[x_t, y_{t+1}] = \nu(x_t) - \nu(y_{t+1})$ , take  $y_t \in U_t$  such that  $y_t > x_t$ , take  $x_{t-1} \in D_{t-1}$  such that  $x_{t-1} < y_t$  and  $\text{rank}[x_{t-1}, y_t] = \nu(x_{t-1}) - \nu(y_t)$ , take  $y_{t-1} \in U_{t-1}$  such that  $y_{t-1} > x_{t-1}$  and so on. Then it is easily verified that  $y_1, x_1, \dots, y_t, x_t$  is a sequence with condition N and

$$\nu(x_i) - \nu(y_{i+1}) = \text{rank}[x_i, y_{i+1}]$$

for  $0 \leq i \leq t$ . ■

Here we state the following fact.



**Lemma 3.4** *There are only finitely many sequences with condition N.*

**Proof** If  $y_1, x_1, \dots, y_t, x_t$  is a sequence with condition N, then  $x_i \leq y_i$  and  $x_j \not\leq y_i$  for  $j > i$ . Therefore, we see that  $x_1, \dots, x_t$  are distinct elements of  $P$ . Since  $P$  is a finite set, we see the result. ■

Now we make the following

**Definition** We set  $r_{\max} := \max\{r(y_1, x_1, \dots, y_t, x_t) \mid y_1, x_1, \dots, y_t, x_t \text{ is a sequence with condition N}\}$ .

We note here the following fact.

**Remark**  $r$  is not necessarily equal to  $\min\{r(y_1, x_1, \dots, y_t, x_t) \mid y_1, x_1, \dots, y_t, x_t \text{ is a sequence with condition N}\}$ .

As a corollary of Lemma 3.3, we see the following fact.

**Corollary 3.5** *If  $\nu$  is a minimal element of  $\mathcal{T}(P)$ , then  $r \leq \nu(x_0) \leq r_{\max}$ .*

**Proof** Since  $\nu \in \mathcal{T}(P)$ , we see that  $r \leq \nu(x_0)$ . On the other hand, by Lemma 3.3, we see that there is a sequence  $y_1, x_1, \dots, y_t, x_t$  with condition N such that  $\nu(x_{i-1}) - \nu(y_i) = \text{rank}[x_{i-1}, y_i]$  for  $1 \leq i \leq t+1$ , where we set  $y_{t+1} = \infty$ . Thus,

$$\begin{aligned}
\nu(x_0) &= \sum_{i=1}^t (\nu(x_{i-1}) - \nu(y_i) + \nu(y_i) - \nu(x_i)) + \nu(x_t) - \nu(y_{t+1}) \\
&= \sum_{i=1}^t (\text{rank}[x_{i-1}, y_i] - (\nu(x_i) - \nu(y_i))) + \text{rank}[x_t, y_{t+1}] \\
&\leq \sum_{i=1}^t (\text{rank}[x_{i-1}, y_i] - \text{rank}[x_i, y_i]) + \text{rank}[x_t, y_{t+1}] \\
&= r(y_1, x_1, \dots, y_t, x_t) \\
&\leq r_{\max}.
\end{aligned}$$

■

Next we define two elements of  $\mathcal{T}(P)$  defined by a sequence of elements with condition N.

**Definition 3.6** Let  $y_1, x_1, \dots, y_t, x_t$  be a sequence of elements with condition N. Set  $y_{t+1} = \infty$ . We define

$$\mu^\downarrow_{(y_1, x_1, \dots, y_t, x_t)}(y_i) := \sum_{k=i}^t (-\text{rank}[x_k, y_k] + \text{rank}[x_k, y_{k+1}])$$

for  $1 \leq i \leq t+1$  and

$$\nu^\downarrow_{(y_1, x_1, \dots, y_t, x_t)}(z) := \max\{\text{rank}[z, y_i] + \mu^\downarrow_{(y_1, x_1, \dots, y_t, x_t)}(y_i) \mid z \leq y_i\}$$

for  $z \in P^+$ . We also define

$$\mu^\uparrow_{(y_1, x_1, \dots, y_t, x_t)}(x_i) := r_{\max} + \sum_{k=1}^i (-\text{rank}[x_{k-1}, y_k] + \text{rank}[x_k, y_k])$$

for  $0 \leq i \leq t$  and

$$\nu^\uparrow_{(y_1, x_1, \dots, y_t, x_t)}(z) := \min\{-\text{rank}[x_i, z] + \mu^\uparrow_{(y_1, x_1, \dots, y_t, x_t)}(x_i) \mid x_i \leq z\}$$

for  $z \in P^+$ .

Here we note the following fact.

**Lemma 3.7** Let  $y_1, x_1, \dots, y_t, x_t$  be a sequence of elements with condition N. Then  $\nu^\downarrow_{(y_1, x_1, \dots, y_t, x_t)}, \nu^\uparrow_{(y_1, x_1, \dots, y_t, x_t)} \in \mathcal{T}(P)$ .

**Proof** Set  $\nu^\downarrow = \nu^\downarrow_{(y_1, x_1, \dots, y_t, x_t)}$ ,  $\nu^\uparrow = \nu^\uparrow_{(y_1, x_1, \dots, y_t, x_t)}$  and  $y_{t+1} = \infty$ . It is easily verified by the definition that if  $z_1, z_2 \in P^+$  and  $z_1 < z_2$ , then

$$\nu^\downarrow(z_1) > \nu^\downarrow(z_2) \quad \text{and} \quad \nu^\uparrow(z_1) > \nu^\uparrow(z_2).$$

Further, since  $\nu^\downarrow(\infty) = \max\{\text{rank}[\infty, y_i] + \mu^\downarrow_{(y_1, x_1, \dots, y_t, x_t)}(y_i) \mid \infty \leq y_i\} = 0$ , we see by the above inequality that  $\nu^\downarrow(z) > 0$  for any  $z \in P$ .

Now we prove that  $\nu^\uparrow(z) > 0$  for any  $z \in P$ . It is enough to show that  $\nu^\uparrow(\infty) = \min\{-\text{rank}[x_i, \infty] + \mu^\uparrow_{(y_1, x_1, \dots, y_t, x_t)}(x_i) \mid x_i \leq \infty\} \geq 0$ . Assume the contrary and take  $i$  with  $-\text{rank}[x_i, \infty] + \mu^\uparrow_{(y_1, x_1, \dots, y_t, x_t)}(x_i) < 0$ . Then  $y_1, x_1, \dots, y_i, x_i$  is a sequence with condition N and  $r_{\max} - r(y_1, x_1, \dots, y_i, x_i) = -\text{rank}[x_i, \infty] + \mu^\uparrow_{(y_1, x_1, \dots, y_t, x_t)}(x_i) < 0$ . This contradicts to the definition of  $r_{\max}$ . ■

We state the following important properties of  $\nu^\downarrow$  and  $\nu^\uparrow$ .

**Lemma 3.8** *Let  $y_1, x_1, \dots, y_t, x_t$  be a sequence of elements in  $P$  with condition N. Suppose that  $r(y_1, x_1, \dots, y_t, x_t) = r_{\max}$ . Then  $\nu^\downarrow_{(y_1, x_1, \dots, y_t, x_t)}$  and  $\nu^\uparrow_{(y_1, x_1, \dots, y_t, x_t)}$  are minimal elements of  $\mathcal{T}(P)$ . Further,*

$$\nu^\downarrow_{(y_1, x_1, \dots, y_t, x_t)}(y_i) = \mu^\downarrow_{(y_1, x_1, \dots, y_t, x_t)}(y_i), \quad (3.1)$$

$$\nu^\downarrow_{(y_1, x_1, \dots, y_t, x_t)}(x_{i-1}) = \text{rank}[x_{i-1}, y_i] + \mu^\downarrow_{(y_1, x_1, \dots, y_t, x_t)}(y_i), \quad (3.2)$$

$$\nu^\uparrow_{(y_1, x_1, \dots, y_t, x_t)}(x_{i-1}) = \mu^\uparrow_{(y_1, x_1, \dots, y_t, x_t)}(x_{i-1}), \quad (3.3)$$

$$\nu^\uparrow_{(y_1, x_1, \dots, y_t, x_t)}(y_i) = -\text{rank}[x_{i-1}, y_i] + \mu^\uparrow_{(y_1, x_1, \dots, y_t, x_t)}(x_{i-1}) \quad (3.4)$$

for  $1 \leq i \leq t+1$ , where we set  $y_{t+1} = \infty$ . In particular,  $\nu^\downarrow_{(y_1, x_1, \dots, y_t, x_t)}(x_0) = \nu^\uparrow_{(y_1, x_1, \dots, y_t, x_t)}(x_0) = r_{\max}$ .

**Proof** By Lemma 3.7, we see that  $\nu^\downarrow_{(y_1, x_1, \dots, y_t, x_t)} \in \mathcal{T}(P)$ . First we show (3.1). Assume the contrary and take  $j$  with  $\nu^\downarrow_{(y_1, x_1, \dots, y_t, x_t)}(y_j) \neq \mu^\downarrow_{(y_1, x_1, \dots, y_t, x_t)}(y_j)$ . Then  $j \leq t$ ,  $\nu^\downarrow_{(y_1, x_1, \dots, y_t, x_t)}(y_j) > \mu^\downarrow_{(y_1, x_1, \dots, y_t, x_t)}(y_j)$  and there exists  $y_i$  such that  $y_j \leq y_i$  and

$$\nu^\downarrow_{(y_1, x_1, \dots, y_t, x_t)}(y_j) = \text{rank}[y_j, y_i] + \mu^\downarrow_{(y_1, x_1, \dots, y_t, x_t)}(y_i).$$

Since  $y_1, x_1, \dots, y_t, x_t$  is a sequence with condition N,  $y_k \not\leq x_j$  for any  $k$  with  $1 \leq k \leq j-1$ . Thus,  $i \geq j$ , since  $x_j < y_j \leq y_i$ . Since  $x_{j-1} < y_j \leq y_i$ , we see that  $y_1, x_1, \dots, y_{j-1}, x_{j-1}, y_i, x_i, \dots, y_t, x_t$  is a sequence with condition N. Further, since

$$\text{rank}[x_{j-1}, y_i] \geq \text{rank}[x_{j-1}, y_j] + \text{rank}[y_j, y_i],$$

we see that

$$\begin{aligned} & r(y_1, x_1, \dots, y_{j-1}, x_{j-1}, y_i, x_i, \dots, y_t, x_t) \\ &= \sum_{k=1}^{j-1} (\text{rank}[x_{k-1}, y_k] - \text{rank}[x_k, y_k]) + \text{rank}[x_{j-1}, y_i] + \mu^\downarrow_{(y_1, x_1, \dots, y_t, x_t)}(y_i) \\ &\geq \sum_{k=1}^{j-1} (\text{rank}[x_{k-1}, y_k] - \text{rank}[x_k, y_k]) \\ &\quad + \text{rank}[x_{j-1}, y_j] + \text{rank}[y_j, y_i] + \mu^\downarrow_{(y_1, x_1, \dots, y_t, x_t)}(y_i) \\ &= \sum_{k=1}^{j-1} (\text{rank}[x_{k-1}, y_k] - \text{rank}[x_k, y_k]) + \text{rank}[x_{j-1}, y_j] + \nu^\downarrow_{(y_1, x_1, \dots, y_t, x_t)}(y_j) \\ &> \sum_{k=1}^{j-1} (\text{rank}[x_{k-1}, y_k] - \text{rank}[x_k, y_k]) + \text{rank}[x_{j-1}, y_j] + \mu^\downarrow_{(y_1, x_1, \dots, y_t, x_t)}(y_j) \\ &= r(y_1, x_1, \dots, y_t, x_t) \\ &= r_{\max}. \end{aligned}$$

This contradicts to the definition of  $r_{\max}$ . Therefore, we see (3.1).

Next we show (3.2). Assume the contrary and take  $j$  with  $\nu_{(y_1, x_1, \dots, y_t, x_t)}^\downarrow(x_j) \neq \text{rank}[x_j, y_{j+1}] + \mu_{(y_1, x_1, \dots, y_t, x_t)}^\downarrow(y_{j+1})$ . Then  $\nu_{(y_1, x_1, \dots, y_t, x_t)}^\downarrow(x_j) > \text{rank}[x_j, y_{j+1}] + \mu_{(y_1, x_1, \dots, y_t, x_t)}^\downarrow(y_{j+1})$  and there is  $y_i$  such that  $x_j \leq y_i$  and

$$\nu_{(y_1, x_1, \dots, y_t, x_t)}^\downarrow(x_j) = \text{rank}[x_j, y_i] + \mu_{(y_1, x_1, \dots, y_t, x_t)}^\downarrow(y_i).$$

Since  $y_1, x_1, \dots, y_t, x_t$  is a sequence with condition N, we see that  $i \geq j$ . Moreover, since

$$\begin{aligned} & \text{rank}[x_j, y_j] + \mu_{(y_1, x_1, \dots, y_t, x_t)}^\downarrow(y_j) \\ &= \text{rank}[x_j, y_{j+1}] + \mu_{(y_1, x_1, \dots, y_t, x_t)}^\downarrow(y_{j+1}) \\ &\neq \nu_{(y_1, x_1, \dots, y_t, x_t)}^\downarrow(x_j), \end{aligned}$$

we see that  $i \geq j + 2$ . Therefore,  $y_1, x_1, \dots, y_j, x_j, y_i, x_i, \dots, y_t, x_t$  is a sequence with condition N and

$$\begin{aligned} & r(y_1, x_1, \dots, y_j, x_j, y_i, x_i, \dots, y_t, x_t) \\ &= \sum_{k=1}^j (\text{rank}[x_{k-1}, y_k] - \text{rank}[x_k, y_k]) + \text{rank}[x_j, y_i] + \mu_{(y_1, x_1, \dots, y_t, x_t)}^\downarrow(y_i) \\ &= \sum_{k=1}^j (\text{rank}[x_{k-1}, y_k] - \text{rank}[x_k, y_k]) + \nu_{(y_1, x_1, \dots, y_t, x_t)}^\downarrow(x_j) \\ &> \sum_{k=1}^j (\text{rank}[x_{k-1}, y_k] - \text{rank}[x_k, y_k]) + \text{rank}[x_j, y_{j+1}] + \mu_{(y_1, x_1, \dots, y_t, x_t)}^\downarrow(y_{j+1}) \\ &= r(y_1, x_1, \dots, y_t, x_t) \\ &= r_{\max}, \end{aligned}$$

contradicting the definition of  $r_{\max}$ . Therefore, we see (3.2).

By (3.1), (3.2) and Lemma 3.2, we see that  $\nu_{(y_1, x_1, \dots, y_t, x_t)}^\downarrow$  is a minimal element of  $\mathcal{T}(P)$ . Further, we see that  $\nu_{(y_1, x_1, \dots, y_t, x_t)}^\downarrow(x_0) = r(y_1, x_1, \dots, y_t, x_t) = r_{\max}$ .

By considering the poset  $Q$  whose base set is  $P^+$  and

$$z < w \text{ in } Q \iff z > w \text{ in } P^+,$$

we see that  $\nu_{(y_1, x_1, \dots, y_t, x_t)}^\uparrow$  is also a minimal element of  $\mathcal{T}(P)$  and (3.3) and (3.4) hold. ■

Since  $\deg T^\nu = \nu(x_0)$  for  $\nu \in \mathcal{T}(P)$ , we see by Corollaries 2.6 and 3.5 and Lemma 3.8, the following

**Theorem 3.9**  $\mathcal{R}_K(H)$  is level if and only if  $r(y_1, x_1, \dots, y_t, x_t) \leq r$  for any sequence of elements in  $P$  with condition N, i.e.,  $r_{\max} = r$ .

As a corollary, we can reprove our previous result.

**Corollary 3.10 ([3])** If  $[x, \infty]$  is pure for any  $x \in P \setminus \{x_0\}$ , then  $\mathcal{R}_K(H)$  is level.

**Proof** By assumption,

$$\text{rank}[x, y] = \text{rank}[x, \infty] - \text{rank}[y, \infty]$$

for any  $x, y \in P \setminus \{x_0\}$  with  $x \leq y$ . Therefore, for any sequence  $y_1, x_1, \dots, y_t, x_t$  with condition N,

$$\begin{aligned} & r(y_1, x_1, \dots, y_t, x_t) \\ &= \sum_{i=1}^t (\text{rank}[x_{i-1}, y_i] - \text{rank}[x_i, y_i]) + \text{rank}[x_t, \infty] \\ &= \text{rank}[x_0, y_1] + \text{rank}[y_1, \infty] \\ &\leq r. \end{aligned}$$

■

Next we show that for any integer  $d$  with  $r \leq d \leq r_{\max}$ , there is a generator of the canonical module of  $\mathcal{R}_K(H)$  with degree  $d$ . First we state the following

**Lemma 3.11** Let  $\nu$  be a minimal element of  $\mathcal{T}(P)$  and  $k$  a positive integer. Set  $\nu_k(x) = \max\{\nu(x) - k, \text{rank}[x, \infty]\}$  for  $x \in P^+$ . Then  $\nu_k$  is a minimal element of  $\mathcal{T}(P)$ .

**Proof** It is clear that  $\nu_k \in \mathcal{T}(P)$ . By Lemma 3.3, we see that there is a sequence  $y_1, x_1, \dots, y_t, x_t$  of elements of  $P$  with condition N such that

$$\nu(x_i) - \nu(y_{i+1}) = \text{rank}[x_i, y_{i+1}]$$

for  $0 \leq i \leq t$ , where we set  $y_{t+1} = \infty$ .

First consider the case where  $\nu_k(x_i) - \nu_k(y_{i+1}) = \text{rank}[x_i, y_{i+1}]$  for  $0 \leq i \leq t$ . Then, by Lemma 3.2, we see that  $\nu_k$  is a minimal element of  $\mathcal{T}(P)$ . Next

consider the case that there exists  $i$  with  $\nu_k(x_i) - \nu_k(y_{i+1}) \neq \text{rank}[x_i, y_{i+1}]$ . Set

$$j = \min\{i \mid \nu_k(x_i) - \nu_k(y_{i+1}) > \text{rank}[x_i, y_{i+1}]\}.$$

If  $\nu_k(x_j) = \nu(x_j) - k$ , then

$$\begin{aligned} & \nu_k(x_j) - \nu_k(y_{j+1}) \\ &= \nu(x_j) - k - \max\{\nu(y_{j+1}) - k, \text{rank}[y_{j+1}, \infty]\} \\ &\leq \nu(x_j) - \nu(y_{j+1}) \\ &= \text{rank}[x_j, y_{j+1}]. \end{aligned}$$

This contradicts to the definition of  $j$ .

Thus,  $\nu_k(x_j) = \text{rank}[x_j, \infty]$ . Since  $\nu_k(x_i) - \nu_k(y_{i+1}) = \text{rank}[x_i, y_{i+1}]$  for  $0 \leq i \leq j-1$  by the definition of  $j$ , by applying Lemma 3.2 to  $y_1, x_1, \dots, y_j, x_j$ , we see that  $\nu_k$  is a minimal element of  $\mathcal{T}(P)$ . ■

As a corollary, we see the following fact.

**Theorem 3.12** *Let  $d$  be an integer with  $r \leq d \leq r_{\max}$ . Then there exists a generator of the canonical module of  $\mathcal{R}_K(H)$  with degree  $d$ .*

**Proof** Set  $k = r_{\max} - d$ . Take a sequence  $y_1, x_1, \dots, y_t, x_t$  with condition N such that  $r(y_1, x_1, \dots, y_t, x_t) = r_{\max}$  and put

$$\nu(z) = \max\{\nu^\downarrow_{(y_1, x_1, \dots, y_t, x_t)}(z) - k, \text{rank}[z, \infty]\}$$

for  $z \in P$ . Then  $\nu(x_0) = \max\{\nu^\downarrow_{(y_1, x_1, \dots, y_t, x_t)}(x_0) - k, r\} = \max\{r_{\max} - k, r\} = \max\{d, r\} = d$ . Since  $\nu$  is a minimal element of  $\mathcal{T}(P)$  by Lemmas 3.11 and 3.8, we see by Corollary 2.4 that  $T^\nu$  is a generator of the canonical module of  $\mathcal{R}_K(H)$  with degree  $d$ . ■

Finally in this section, we make a remark on  $P$  when  $\mathcal{R}_K(H)$  is level. First we state the following

**Lemma 3.13** *Set  $F = \{x \in P \mid \text{rank}[x_0, x]_{P^+} + \text{rank}[x, \infty]_{P^+} < r\}$ . Suppose that  $\mathcal{R}_K(H)$  is level. Then for any  $z_1, z_2 \in P^+ \setminus F$  with  $z_1 \leq z_2$ ,*

$$\text{rank}[x_0, z_1]_{P^+} + \text{rank}[z_1, z_2]_{P^+} + \text{rank}[z_2, \infty]_{P^+} = r. \quad (3.5)$$

and

$$\text{rank}[z_1, z_2]_{P^+ \setminus F} = \text{rank}[z_1, z_2]_{P^+}. \quad (3.6)$$

**Proof** We first show (3.5). The cases where  $z_1 = x_0$ ,  $z_2 = \infty$  or  $z_1 = z_2$  are clear from the definition of  $F$ . Suppose that  $z_1, z_2 \in P \setminus F$  and  $x_0 < z_1 < z_2$ . Then  $z_2, z_1$  is a sequence with condition N. Since  $\mathcal{R}_K(H)$  is level, we see by Theorem 3.9 that

$$\text{rank}[x_0, z_2]_{P^+} - \text{rank}[z_1, z_2]_{P^+} + \text{rank}[z_1, \infty]_{P^+} \leq r.$$

Since  $z_1, z_2 \notin F$ , we see that  $\text{rank}[x_0, z_2]_{P^+} = r - \text{rank}[z_2, \infty]_{P^+}$  and  $\text{rank}[z_1, \infty]_{P^+} = r - \text{rank}[x_0, z_1]_{P^+}$ . Thus,

$$2r - \text{rank}[x_0, z_1]_{P^+} - \text{rank}[z_1, z_2]_{P^+} - \text{rank}[z_2, \infty]_{P^+} \leq r,$$

i.e.,

$$\text{rank}[x_0, z_1]_{P^+} + \text{rank}[z_1, z_2]_{P^+} + \text{rank}[z_2, \infty]_{P^+} \geq r.$$

The opposite inequality is obvious. Thus, we see (3.5).

Next we prove (3.6). Assume the contrary. Take a maximal chain

$$z_1 = w_0 < w_1 < \cdots < w_t = z_2$$

of  $[z_1, z_2]_{P^+}$  such that  $t = \text{rank}[z_1, z_2]_{P^+}$ . Since

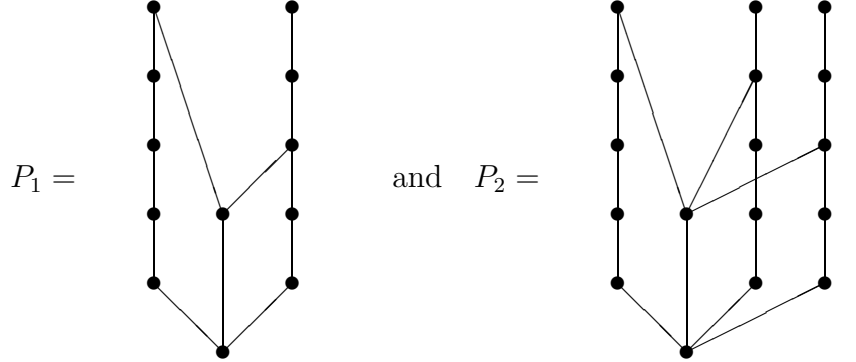
$$\begin{aligned} & \text{rank}[x_0, w_i]_{P^+} + \text{rank}[w_i, \infty]_{P^+} \\ & \geq \text{rank}[x_0, z_1]_{P^+} + \text{rank}[z_1, w_i]_{P^+} + \text{rank}[w_i, z_2]_{P^+} + \text{rank}[z_2, \infty]_{P^+} \\ & = \text{rank}[x_0, z_1]_{P^+} + \text{rank}[z_1, z_2]_{P^+} + \text{rank}[z_2, \infty]_{P^+} \\ & = r, \end{aligned}$$

we see that  $w_i \notin F$  for any  $i = 1, \dots, t-1$ . Therefore,  $\text{rank}[z_1, z_2]_{P^+ \setminus F} \geq t = \text{rank}[z_1, z_2]_{P^+}$ . The opposite inequality is obvious. ■

By the above lemma and Lemma 2.9, we see the following fact.

**Proposition 3.14** *Let  $F$  be as in Lemma 3.13. If  $\mathcal{R}_K(H)$  is level, then  $P^+ \setminus F$  is pure of rank  $r$ . In particular, if  $\mathcal{R}_K(H)$  is level and  $F = \emptyset$ , then  $\mathcal{R}_K(H)$  is Gorenstein.*

**Example 3.15** Let



Then  $\text{rank} P_i^+ = 6$  for  $i = 1, 2$ . Let  $F_i$  be the subset of  $P_i$  defined as in Lemma 3.13 and let  $H_i$  be the distributive lattice corresponding to  $P_i$  for  $i = 1, 2$ . Then  $P_i^+ \setminus F_i$  is pure but  $\mathcal{R}_K(H_i)$  is not level for  $i = 1, 2$ . Thus, the converse of Proposition 3.14 does not hold. There are 2 generators with degree 6, 3 generators with degree 7 and 6 generators with degree 8 of the canonical module of  $\mathcal{R}_K(H_1)$  and there are 2 generators with degree 6, 48 generators with degree 7 and 108 generators with degree 8 of the canonical module of  $\mathcal{R}_K(H_2)$ .

## 4 Characterization of level type 2 Hibi rings

As Example 3.15 shows, it is very hard to describe Cohen-Macaulay type of the Hibi ring in terms of the combinatorial structure of  $P$ . However, we can characterize Hibi ring  $\mathcal{R}_K(H)$  to be of type 2 with respect to the combinatorial property of  $P$ . Recall that, by Corollary 2.4, we see that  $\text{type} \mathcal{R}_K(H)$  is the number of minimal elements of  $\mathcal{T}(P)$ .

In this section, we state a characterization of a Hibi ring to be level and of type 2. First we make the following

**Definition** Let  $y_1, x_1, y_2, x_2, \dots, y_t, x_t$  be a sequence of elements in  $P$ . If the following 4 conditions are satisfied, we say that  $y_1, x_1, y_2, x_2, \dots, y_t, x_t$  is an irredundant sequence.

- (1)  $y_1, x_1, y_2, x_2, \dots, y_t, x_t$  satisfies condition N.
- (2)  $r(y_1, x_1, \dots, y_t, x_t) = r_{\max}$ .



- (3) If  $y'_1, x'_1, y'_2, x'_2, \dots, y'_{t'}, x'_{t'}$  is a sequence of elements in  $P$  with condition N and  $r(y'_1, x'_1, \dots, y'_{t'}, x'_{t'}) = r_{\max}$ , then  $t \leq t'$ .
- (4) For any  $i$  with  $1 \leq i \leq t$ ,  $\text{rank}[z, y_{i+1}] - \text{rank}[z, y_i] < \text{rank}[x_i, y_{i+1}] - \text{rank}[x_i, y_i]$  for any  $z \in (x_i, y_{i+1}] \cap (x_i, y_i]$  and  $\text{rank}[x_{i-1}, z] - \text{rank}[x_i, z] < \text{rank}[x_{i-1}, y_i] - \text{rank}[x_i, y_i]$  for any  $z \in [x_{i-1}, y_i) \cap [x_i, y_i)$ , where we set  $y_{t+1} = \infty$ .

It is clear that there exists an irredundant sequence. Further, by Theorem 3.9,  $\mathcal{R}_K(H)$  is level if and only if the empty sequence is an irredundant sequence.

Next we state the following

**Lemma 4.1** *Set  $F = \{x \in P \mid \text{rank}[x_0, x] + \text{rank}[x, \infty] < r\}$ . Then the number of generators of the canonical module of degree  $r$  is greater than  $\#F$ . In particular,  $\text{type}\mathcal{R}_K(H) > \#F$ .*

**Proof** Set  $F = \{f_1, f_2, \dots, f_u\}$  and  $i < j$  if  $f_i < f_j$ . For  $t$  with  $1 \leq t \leq u+1$ , set

$$\nu_t(x) = \begin{cases} \text{rank}[x, \infty] & \text{if } x \notin F \text{ or } x \in F \text{ and } x = f_j \text{ with } j \geq t, \\ r - \text{rank}[x_0, x] & \text{otherwise.} \end{cases}$$

Then it is easily verified that  $\nu_t$  is an element of  $\mathcal{T}(P)$ . Further, since  $\nu_t(x_0) = r$ , we see that  $\nu_t$  is a minimal element of  $\mathcal{T}(P)$ .

Since for any  $t, t'$  with  $t < t'$ ,  $\nu_t(f_t) < \nu_{t'}(f_t)$ , we see that  $\nu_t \neq \nu_{t'}$ . Therefore, we see that there are at least  $u+1$  minimal elements  $\nu$  of  $\mathcal{T}(P)$  such that  $\nu(x_0) = r$ . ■

Now we state the main theorem of this section.

**Theorem 4.2**  *$\mathcal{R}_K(H)$  is level and  $\text{type}\mathcal{R}_K(H) = 2$  if and only if there exists  $z \in P$  with the following conditions.*

- (1)  $\text{rank}[x_0, z] + \text{rank}[z, \infty] = r - 1$ .
- (2)  $P^+ \setminus \{z\}$  is pure of rank  $r$ .
- (3)  $[x_0, z]$  and  $[z, \infty]$  are pure.

**Remark** As Example 3.15 shows, (3) of Theorem 4.2 does not follow from (1) and (2).

**Proof of Theorem 4.2** We first assume that  $\mathcal{R}_K(H)$  is level and  $\text{type}\mathcal{R}_K(H) = 2$  and prove that there exists  $z \in P$  with (1), (2) and (3). Set  $F = \{x \in P \mid \text{rank}[x_0, x] + \text{rank}[x, \infty] < r\}$ . If  $F = \emptyset$  then, by Proposition 3.14, we see that  $\mathcal{R}_K(H)$  is Gorenstein, contradicting the assumption. Therefore,  $F \neq \emptyset$ . If  $\#F \geq 2$ , then by Lemma 4.1, we see that  $\text{type}\mathcal{R}_K(H) > 2$ , again contradicting the assumption. Thus,  $\#F = 1$ .

Set  $F = \{z\}$ . We show that  $z$  satisfies (1), (2) and (3). Suppose that

$$\text{rank}[x_0, z] + \text{rank}[z, \infty] \leq r - 2.$$

Then if we set

$$\mu_t(x) = \begin{cases} \text{rank}[x, \infty] & \text{if } x \neq z, \\ \text{rank}[x, \infty] + t & \text{if } x = z, \end{cases}$$

then  $\mu_t$  is a minimal element of  $\mathcal{T}(P)$  for  $0 \leq t \leq 2$ . Thus,  $\text{type}\mathcal{R}_K(H) \geq 3$ , contradicting the assumption.

Therefore,

$$\text{rank}[x_0, z] + \text{rank}[z, \infty] = r - 1,$$

i.e., we see (1). Further, we see by Proposition 3.14, that  $P^+ \setminus \{z\}$  is pure of rank  $r$ . Thus, we see (2).

Now let  $y$  be an arbitrary element of  $[z, \infty]$  such that  $z < y$ . We shall show that  $[y, \infty]$  is pure and  $\text{rank}[y, \infty] = \text{rank}[z, \infty] - 1$ . The case where  $y = \infty$  is clear. Suppose  $y \neq \infty$ . Then  $y, z$  is a sequence with condition N. Since  $\mathcal{R}_K(H)$  is level, we see by Theorem 3.9 that  $\text{rank}[x_0, y] - 1 + \text{rank}[z, \infty] = r(y, z) \leq r$ . Since  $\text{rank}[x_0, y] = r - \text{rank}[y, \infty]$ , we see that  $\text{rank}[z, \infty] - \text{rank}[y, \infty] \leq 1$ . Therefore,  $\text{rank}[y, \infty] = \text{rank}[z, \infty] - 1$ . Further,  $[y, \infty]$  is pure by Proposition 3.14.

Since  $y$  is an arbitrary element of  $[z, \infty]$  with  $z < y$ , we see that  $[z, \infty]$  is pure. We see that  $[x_0, z]$  is pure by the same way. Thus, we see (3).

Next we assume that there exists  $z \in P$  with (1), (2) and (3) and prove that  $\mathcal{R}_K(H)$  is level and  $\text{type}\mathcal{R}_K(H) = 2$ .

We first note that it follows from (2) that for any  $w_1, w_2 \in P^+ \setminus \{z\}$ ,  $\text{rank}[w_1, w_2]_{P^+ \setminus \{z\}} = \text{rank}[w_1, w_2]_{P^+}$ . In particular,  $\text{rank}[x, y] = \text{rank}[x_0, y] - \text{rank}[x_0, x]$  for any  $x, y \in P^+ \setminus \{z\}$  with  $x < y$ . We also see that if  $x < z$ , then  $\text{rank}[x, z] = \text{rank}[x_0, z] - \text{rank}[x_0, x]$  and if  $y > z$ , then  $\text{rank}[z, y] = \text{rank}[z, \infty] - \text{rank}[y, \infty]$ , since  $[x_0, z]$  and  $[z, \infty]$  are pure.

Now let  $y_1, x_1, \dots, y_t, x_t$  be an irredundant sequence and set  $y_{t+1} = \infty$ . We shall show that  $t = 0$ . Assume the contrary. Then, since  $y_1, x_1, \dots, y_t, x_t$  is an irredundant sequence, we see that  $r(y_2, x_2, \dots, y_t, x_t) < r(y_1, x_1, \dots, y_t, x_t)$ , i.e.,

$$\text{rank}[x_0, y_2] < \text{rank}[x_0, y_1] - \text{rank}[x_1, y_1] + \text{rank}[x_1, y_2]. \quad (4.1)$$

First consider the case where  $x_1 \neq z$ . Since  $\text{rank}[x_1, y_i] = \text{rank}[x_0, y_i] - \text{rank}[x_0, x_1]$  for  $i = 1, 2$ , we see that the right hand side of (4.1) is equal to  $\text{rank}[x_0, y_2]$ . This is a contradiction.

Next consider the case where  $x_1 = z$ . Since  $\text{rank}[x_1, y_i] = \text{rank}[x_1, \infty] - \text{rank}[y_i, \infty]$  for  $i = 1, 2$  we see that the right hand side of (4.1) is equal to

$$\text{rank}[x_0, y_1] + \text{rank}[y_1, \infty] - \text{rank}[y_2, \infty].$$

Since  $y_i \neq z$  for  $i = 1, 2$ , we see that  $\text{rank}[y_i, \infty] = r - \text{rank}[x_0, y_i]$  for  $i = 1, 2$ . Therefore, the right hand side of (4.1) is equal to  $\text{rank}[x_0, y_2]$ . This is also a contradiction. Thus, we see that  $t = 0$  and  $\mathcal{R}_K(H)$  is level.

Let  $\nu$  be an arbitrary minimal element of  $\mathcal{T}(P)$ . Since  $\mathcal{R}_K(H)$  is level, we see that  $\nu(x_0) = r$ . Thus, we see that  $\nu(x) \geq \text{rank}[x, \infty]$  and  $r - \nu(x) = \nu(x_0) - \nu(x) \geq \text{rank}[x_0, x]$  i.e.,  $\text{rank}[x, \infty] \leq \nu(x) \leq r - \text{rank}[x_0, x]$  for any  $x \in P$ . In particular, we see by (2) that  $\nu(x) = \text{rank}[x, \infty]$  for any  $x \in P^+ \setminus \{z\}$ . We also see by (1) that

$$\text{rank}[z, \infty] \leq \nu(z) \leq r - \text{rank}[x_0, z] = \text{rank}[z, \infty] + 1.$$

Thus  $\text{type}\mathcal{R}_K(H) \leq 2$ . Further, we see  $\text{type}\mathcal{R}_K(H) \geq 2$  by Lemma 4.1. ■

## 5 Characterization of non-level type 2 Hibi rings

In this final section, we give a characterization of a Hibi ring to be non-level and of type 2.

**Theorem 5.1**  *$\mathcal{R}_K(H)$  is non-level and  $\text{type}\mathcal{R}_K(H) = 2$  if and only if there exist  $x, y \in P \setminus \{x_0\}$  with the following conditions.*

- (1)  $x < y$ .
- (2)  $\text{rank}[x_0, y] + \text{rank}[x, \infty] = r + 2$ .
- (3)  $P^+ = [x_0, y] \cup [x, \infty]$ .
- (4)  $\text{rank}[x_0, z_1] + \text{rank}[z_1, z_2] + \text{rank}[z_2, \infty] = r$  for any  $z_1, z_2 \in P^+$  with  $z_1 \leq z_2$  and  $(z_1, z_2) \neq (x, y)$ .

**Proof** Set  $F = \{z \in P \mid \text{rank}[x_0, z] + \text{rank}[z, \infty] < r\}$ .

First we assume that  $\mathcal{R}_K(H)$  is non-level and  $\text{type}\mathcal{R}_K(H) = 2$  and prove that there exist  $x, y \in P$  with conditions (1) to (4). Since  $\mathcal{R}_K(H)$  is not

level, we see by Theorem 3.9 that  $r_{\max} > r$ . Further, since  $\text{type}\mathcal{R}_K(H) \geq r_{\max} - r + 1$  by Theorem 3.12, we see that  $r_{\max} = r + 1$  and the generating system of the canonical module of  $\mathcal{R}_K(H)$  consists of 2 elements: one has degree  $r$  and the other one has degree  $r + 1$ . Thus, by Lemma 4.1, we see that  $F = \emptyset$ .

Let  $y_1, x_1, \dots, y_t, x_t$  be an irredundant sequence and set  $y_{t+1} = \infty$ . First we claim that  $t = 1$ . Since  $r_{\max} > r$ , we see that  $t \geq 1$ . Further, since  $y_1, x_1, \dots, y_t, x_t$  is an irredundant sequence, we see that  $r(y_2, x_2, \dots, y_t, x_t) < r(y_1, x_1, \dots, y_t, x_t)$ , i.e.,

$$\text{rank}[x_0, y_2] < \text{rank}[x_0, y_1] - \text{rank}[x_1, y_1] + \text{rank}[x_1, y_2].$$

Moreover, since  $F = \emptyset$ , we see that

$$\begin{aligned} r &= \text{rank}[x_0, y_2] + \text{rank}[y_2, \infty] \\ &< \text{rank}[x_0, y_1] - \text{rank}[x_1, y_1] + \text{rank}[x_1, y_2] + \text{rank}[y_2, \infty] \\ &\leq \text{rank}[x_0, y_1] - \text{rank}[x_1, y_1] + \text{rank}[x_1, \infty] \\ &= r(y_1, x_1). \end{aligned}$$

Since  $r_{\max} = r + 1$  and  $y_1, x_1, \dots, y_t, x_t$  is an irredundant sequence, we see that  $r(y_1, x_1) = r + 1$  and  $t = 1$ .

Set  $x = x_1$  and  $y = y_1$ . We show that these  $x$  and  $y$  satisfy (1) to (4). We first show that (1). Assume the contrary. Then there is  $w \in P$  such that  $x < w < y$  and  $\text{rank}[x, y] = \text{rank}[x, w] + \text{rank}[w, y]$ . Since  $y, x$  is an irredundant sequence, we see that

$$\text{rank}[x_0, w] - \text{rank}[x, w] \leq \text{rank}[x_0, y] - \text{rank}[x, y] - 1$$

and

$$-\text{rank}[w, y] + \text{rank}[w, \infty] \leq -\text{rank}[x, y] + \text{rank}[x, \infty] - 1.$$

Therefore,

$$\begin{aligned} &\text{rank}[x_0, w] - \text{rank}[x, w] - \text{rank}[w, y] + \text{rank}[w, \infty] \\ &\leq \text{rank}[x_0, y] - 2\text{rank}[x, y] + \text{rank}[x, \infty] - 2 \\ &= r(y, x) - \text{rank}[x, y] - 2. \end{aligned}$$

On the other hand, since

$$\text{rank}[x, w] + \text{rank}[w, y] = \text{rank}[x, y] \quad \text{and} \quad \text{rank}[x_0, w] + \text{rank}[w, \infty] = r,$$

we see  $r \leq r(y, x) - 2$ . This contradicts to the fact that  $r(y, x) = r + 1$ . Therefore, we see (1). Moreover, since

$$\begin{aligned}
& \text{rank}[x_0, y] + \text{rank}[x, \infty] - 1 \\
&= \text{rank}[x_0, y] - \text{rank}[x, y] + \text{rank}[x, \infty] \\
&= r(y, x) \\
&= r + 1,
\end{aligned}$$

we see (2).

Next we prove (3). Assume the contrary and let  $z \in P^+ \setminus ([x_0, y] \cup [x, \infty])$ . Since  $z \not\leq y$ , we see by the definition of  $\nu^\downarrow_{(y,x)}$  (see Definition 3.6) that  $\nu^\downarrow_{(y,x)}(z) = \text{rank}[z, \infty]$ . On the other hand, since  $z \not\leq x$ , we see by the definition of  $\nu^\uparrow_{(y,x)}$  that  $\nu^\uparrow_{(y,x)}(z) = r + 1 - \text{rank}[x_0, z]$ . Since the minimal generating system of the canonical module of  $\mathcal{R}_K(H)$  contains only one element with degree  $r + 1$  and  $\nu^\downarrow_{(y,x)}$  and  $\nu^\uparrow_{(y,x)}$  are minimal elements of  $\mathcal{T}(P)$  with degree  $r + 1$  by Lemma 3.8, we see that  $\nu^\downarrow_{(y,x)} = \nu^\uparrow_{(y,x)}$ . Therefore,

$$\text{rank}[z, \infty] = \nu^\downarrow_{(y,x)}(z) = \nu^\uparrow_{(y,x)}(z) = r + 1 - \text{rank}[x_0, z].$$

This contradicts to the fact that  $\text{rank}P^+ = r$ . Therefore, we see (3).

Finally, we prove (4). The case where  $z_1 = z_2$ ,  $z_1 = x_0$  or  $z_2 = \infty$  are clear since  $F = \emptyset$ . Suppose that  $x_0 < z_1 < z_2 < \infty$ . Then  $z_2, z_1$  is a sequence with condition N. If  $r(z_2, z_1) \leq r$ , then

$$\text{rank}[x_0, z_2] - \text{rank}[z_1, z_2] + \text{rank}[z_1, \infty] \leq r.$$

Since  $F = \emptyset$ , we see that

$$2r - \text{rank}[z_2, \infty] - \text{rank}[z_1, z_2] - \text{rank}[x_0, z_1] \leq r.$$

Therefore,

$$\text{rank}[x_0, z_1] + \text{rank}[z_1, z_2] + \text{rank}[z_2, \infty] \geq r.$$

Since the converse inequality holds in general, we see the result.

Now assume that  $r(z_2, z_1) > r$ . Since  $r_{\max} = r + 1$ , we see that  $r(z_2, z_1) = r_{\max} = r + 1$ . Moreover, since the generating system of the canonical module contains exactly one element with degree  $r + 1$ , we see that

$$\nu^\downarrow_{(y,x)} = \nu^\uparrow_{(y,x)} = \nu^\downarrow_{(z_2,z_1)} = \nu^\uparrow_{(z_2,z_1)}$$

by Lemma 3.8. Since  $\nu^\downarrow_{(z_2,z_1)}(y) = \nu^\downarrow_{(y,x)}(y) > \text{rank}[y, \infty]$ , we see that  $y \leq z_2$ . On the other hand, since  $\nu^\downarrow_{(y,x)}(z_2) = \nu^\downarrow_{(z_2,z_1)}(z_2) > \text{rank}[z_2, \infty]$ , we

see that  $z_2 \leq y$ . Therefore, we see that  $z_2 = y$ . We also see that  $z_1 = x$  by the same way. This proves (4).

Next we assume that there exist  $x, y \in P$  with conditions (1) to (4) and prove that  $\mathcal{R}_K(H)$  is non-level and  $\text{type}\mathcal{R}_K(H) = 2$ . First we see by (4) that  $F = \emptyset$ . Further, we see that  $y, x$  is a sequence with condition N and by (1) and (2) that

$$r(y, x) = \text{rank}[x_0, y] - \text{rank}[x, y] + \text{rank}[x, \infty] = r + 1.$$

Thus, we see that  $r_{\max} > r$ . In particular,  $\mathcal{R}_K(H)$  is not level by Theorem 3.9.

Let  $y_1, x_1, \dots, y_t, x_t$  be an irredundant sequence. Since  $r_{\max} > r$ , we see that  $t \geq 1$ . First consider the case where there is no  $i$  such that  $(x_i, y_i) = (x, y)$ . Then  $r(y_1, x_1, \dots, y_t, x_t) \leq r$  by (1), (2) and (4), contradicting the fact that  $y_1, x_1, \dots, y_t, x_t$  is an irredundant sequence. Thus, there exists  $i$  such that  $(x_i, y_i) = (x, y)$ . Then by (4), we see that  $r(y_1, x_1, \dots, y_t, x_t) = r(y_i, x_i) = r(y, x) = r + 1$ . Therefore, we see that  $y, x$  is the unique irredundant sequence and  $r_{\max} = r(y, x) = r + 1$ .

Let  $\nu$  be a minimal element of  $\mathcal{T}(P)$  with  $\nu(x_0) = r$ . Since  $\nu(z) \geq \text{rank}[z, \infty]$  and  $r - \nu(z) = \nu(x_0) - \nu(z) \geq \text{rank}[x_0, z]$ , we see that  $\nu(z) = \text{rank}[z, \infty]$  for any  $z \in P^+$ , since  $F = \emptyset$ . Therefore, in the generating system of the canonical module of  $\mathcal{R}_K(H)$ , there exists exactly one element with degree  $r$ .

Let  $\nu$  be a minimal element of  $\mathcal{T}(P)$  with  $\nu(x_0) = r + 1$ . It is enough to show that  $\nu = \nu_{(y, x)}^\downarrow$ . As in the proof of Lemma 3.3, set  $U_1 := \{w \in P^+ \mid \text{rank}[x_0, w]_{P^+} = \nu(x_0) - \nu(w)\}$ ,  $D_1 := \{z \in P^+ \setminus U_1 \mid \exists w \in U_1 \text{ such that } w > z\}$ ,  $U_2 := \{w \in P^+ \setminus (U_1 \cup D_1) \mid \exists z \in D_1 \text{ such that } z < w \text{ and } \text{rank}[z, w]_{P^+} = \nu(z) - \nu(w)\}$ ,  $D_2 := \{z \in P^+ \setminus (U_1 \cup D_1 \cup U_2) \mid \exists w \in U_2 \text{ such that } w > z\}$  and so on. Since  $\nu(x_0) = r + 1$ , we see that  $\infty \notin U_1$ . On the other hand, since

$$\infty \in U_1 \cup D_1 \cup U_2 \cup D_2 \cup \dots,$$

we see that  $D_1 \neq \emptyset$ .

Let  $z_1 \in D_1$  and let  $z_2 \in U_1$  such that  $z_2 > z_1$ . Then

$$\begin{aligned} & \text{rank}[x_0, z_2] \\ &= \nu(x_0) - \nu(z_2) \\ &= \nu(x_0) - \nu(z_1) + \nu(z_1) - \nu(z_2) \\ &\geq \text{rank}[x_0, z_1] + 1 + \text{rank}[z_1, z_2]. \end{aligned} \tag{5.1}$$

Therefore, we see by (4) that  $(z_1, z_2) = (x, y)$ . Further, by (1), (2) and the fact that  $F = \emptyset$ , we see that equality holds for (5.1). Therefore, we see that

$$\nu(x_0) - \nu(x) = \text{rank}[x_0, x] + 1 \tag{5.2}$$

and

$$\nu(x) - \nu(y) = 1. \quad (5.3)$$

Let  $z$  be an arbitrary element of  $P$ . We shall show that  $\nu(z) = \nu^\downarrow_{(y,x)}(z)$ .

First consider the case where  $z = x$ . Since  $\nu(x) = \text{rank}[x, \infty]$  by (5.2) and the facts that  $F = \emptyset$  and  $\nu(x_0) = r + 1$ , we see that  $\nu(x) = \nu^\downarrow_{(y,x)}(x)$  by Lemma 3.8.

Next consider the case where  $z = y$ . Since  $\nu(x) = \text{rank}[x, \infty]$ , we see by (5.3) and (1) that  $\nu(y) = \text{rank}[x, \infty] - \text{rank}[x, y] = \nu^\downarrow_{(y,x)}(y)$  by Lemma 3.8.

Now consider the case where  $z \leq y$  and  $z \neq x$ . Since  $\text{rank}[x_0, y] = \text{rank}[x_0, z] + \text{rank}[z, y]$  by (4), we see that  $z \in U_1$  and  $\nu(z) - \nu(y) = \text{rank}[z, y]$ . Since  $\nu(x_0) = r + 1$ , we see that  $\nu(z) = \nu(x_0) - \text{rank}[x_0, z] > \text{rank}[z, \infty]$ . Therefore, we see by the definition of  $\nu^\downarrow_{(y,x)}$  that

$$\nu(z) = \nu(y) + \text{rank}[z, y] = \nu^\downarrow_{(y,x)}(y) + \text{rank}[z, y] = \nu^\downarrow_{(y,x)}(z).$$

Finally consider the case where  $z \not\leq y$ . By (3), we see that  $z > x$ . Further, we see by (4) that  $\text{rank}[x, \infty] = \text{rank}[x, z] + \text{rank}[z, \infty]$ . Since  $\nu(x) = \text{rank}[x, \infty]$ , we see that  $\nu(z) = \text{rank}[z, \infty] = \nu^\downarrow_{(y,x)}(z)$ . ■

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